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Turbulent velocity profiles from stability criteria

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A velocity-defect law for channel flow is shown to result from the single requirement of Reynolds-stress spectral 'smoothness' for any mean profile maintained free of inflexions by transient instabilities. The deduced velocity is a logarithmic function of position near boundaries and parabolic in mid-regions of the flow, independent of the detailed mechanisms of momentum transport by the fluid. However, if the 'smoothness' of the spectral tail is decreased, a second logarithmic layer of steeper slope emerges inside the first layer. Profile data from drag-reduction experiments gathered by Virk (1975) exhibit the deduced inner log layer and its transition region to the usual outer flow.

1. Introduction

Recent experimental and theoretical studies of the infrequent but violent momentum transferring processes in shear boundary layers renew the old hope that a mechanistic rather than statistical understanding of turbulent macro-equilibration is possible. In addition, new data on chemical additives which alter the turbulent boundary layer of shear flows suggest that it will be possible to assess critically the depth of that mechanistic understanding.

This paper was motivated in large part by Landahl's (1975) theoretical interpretation of the brief boundary-layer bursts as 'secondary instabilities' on transient inflexions which are observed in the mean velocity profile. It is not at all clear how such instabilities maintain the mean properties of the flow, nor how those mean properties depend upon the complicated physical and chemical variations of the flow near the boundary. However, it is observed that the resulting mean flows have no inflexions, supporting the classical view of the destabilizing mechanism.

Turbulent Poiseuille flow in channels has been and remains a central object of theoretical study in fluid dynamics. It is used as the first mathematical example in this paper because it is statistically stable, homogeneous in two dimensions, and geometrically simple. Among the earliest observations was the relation between locally averaged flow properties and the stress on the boundaries. Principal among these is the velocity-defect law that, away from the boundaries, the departure of the average flow from its maximum value appears to become independent of viscosity and can be represented quite accurately as

$$\frac{U_m - U}{U_\tau} = G\left(\frac{z}{z_0}\right), \tag{1.1}$$

where U is the average velocity at a distance z from the boundary, U_m is the maximum velocity, z_0 is the half-width of the channel, $U_\tau = \tau_0^{\frac{1}{2}}$ is called the friction velocity, where τ_0 is the wall stress per unit mass, and $G(z/z_0)$ is some universal function of its argument. The report which follows seeks to link the recent mechanistic inquiry with the early statistical observations by deducing the qualitative form of $G(z/z_0)$, in (1.1), both for normal Newtonian flow and for the non-Newtonian flows caused by chemical additives.

Although no deductive theory has successfully predicted the full structure of G, Millikan (1938) has shown that it will have a logarithmc form in any region of overlap it may have with the observed 'law of the wall'. The latter observed law is that near a smooth wall the average velocity appears to become independent of z_0 and can be represented quite accurately as

$$U/U_\tau = F(U_\tau z/\nu), \tag{1.2}$$

where ν is the kinematic viscosity of the fluid and F is some universal function of its argument. For a 'rough' wall ν/U_τ is replaced in the argument of F by an equivalent roughness length. Many people have written on this topic, for there is an exciting element of black magic in general results which are independent of the mechanism (e.g. Yajnik 1970; also Fendell 1972). Yet only a few of these writers have noted that an overlap of (1.1) and (1.2) is not required by either logic or dynamics. The empirical fact is the observation of an (extensive) logarithmic region. It is not a deduction but these observations which establish overlap of the 'laws' (1.1) and (1.2). However, it will be shown here with less recourse to hypothesis that a different, but compatible, explanation of the logarithmic region of the velocity-defect law is possible. Also, this more mechanistic explanation determines the entire form of G.

Drag-reducing chemical additives have provided a new challenge to accepted rationalizations of turbulent shearing flow since they modify mean velocity profiles. Hence we may now establish certain limits of validity of our current understanding, as well as testing new proposals concerning the dissipation mechanisms in turbulence. Virk (1975) has assembled his and others' work in a review, covering a wide variety of measurements with many different chemicals, which are summarized in his idealized schematic diagram (figure 1). Data supporting this idealization are reproduced later in this paper. The principal conclusions drawn by Virk are that point C in figure 1 is determined by the kind and amount of drag-reducing additive used, while the 'ultimate' profile can occupy the entire flow and is independent of the kind of additive used. A first explanation of the 'inner' logarithmic layer is given in §4 here. The implication is that the inner layer is a normal fluid property, not so much created as 'exposed' and extended by the chemical additives.

The observations by Kline *et al.* (1967) of infrequent violent disturbances near boundaries in turbulent shearing flow were among the first to stimulate the current 'mechanistic' era in the study of turbulence. These disturbances appear to be responsible for most of the momentum transport from the fluid to the boundary, and hence are the important processes to study for an understanding of the mean profile and drag reduction. Landahl (1972, 1975, 1977) has explored the interpretation that these disturbances are instabilities occurring on, and moving with, transient inflexions in the mean profile. Presumably, then, instabilities remove inflexions from the mean flow, the mean being established by momentum transfer due to these recurring transient instabilities.

Earlier attempts at quantitative shear-flow turbulence theory based on stability arguments (e.g. Malkus 1954, 1956, 1961; Gol'dshtik 1969; Gol'dshtik *et al.* 1971) had

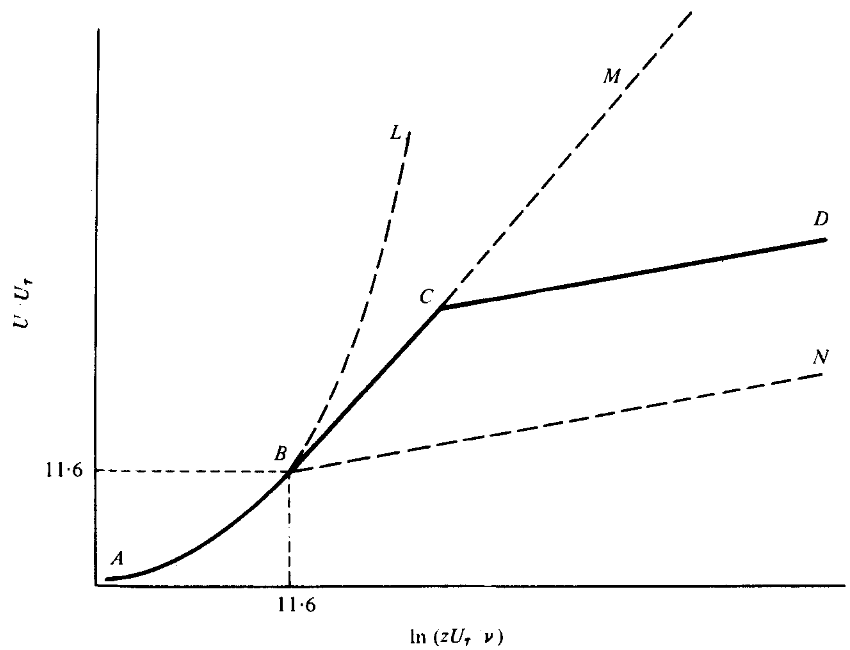

FIGURE 1. An idealized drag-reduction velocity profile (*ABCD*). *BL* is the continuation of the viscous boundary region. *BN* is the normal logarithmic law. *BC* is the 'inner' logarithmic law. *BM* is the 'ultimate' profile. (After Virk 1975.)

explored the possibility that the linear stability of the evolved mean flow was central in determining the turbulent amplitudes. This mechanism was found to be quantitatively inadequate by the careful numerical work of Reynolds & Tiederman (1967) on the linear problem. Their results require that instabilities associated with finite amplitude departures from the mean be responsible for the turbulent momentum transfer, in keeping with the current work described in the last paragraph. However, from the early studies of the role of linear stability in turbulent flow grew the search for formal upper bounds on momentum and heat transport. That profound work, first by Howard (1963, 1972) and then by Busse (1969, 1970), Chan (1971) and Joseph (1976), represents the first completely deductive quantitative results on turbulent flow. Unfortunately, these upper bounds on transport are found to be close to reality in only a few circumstances. A likely reason for this inadequacy emerges in §3. There it is seen that a lack of any formal constraint reflecting the physics of inflexional instability may be the major weakness of the upper-bound theory when applied to shear flows.

The central purpose of this paper is to explore the simplest amechanistic properties of a mean velocity profile maintained by inflexional instabilities to be of positive (or negative) curvature. Section 2 contains the parameter definitions for channel flow and the descriptive formalism for a general representation of everywhere positive functions. Section 3 contains the hypothesis of spectral 'smoothness' in its mathematical form and also the deduced (Newtonian) velocity-defect profile. Section 4 contains examples of the effects of (non-Newtonian) reductions in the spectral tail of the momentum transport. An 'inner' logarithmic layer and its transition region are found. Section 5 contains an application of the method to plane Couette flow and cylindrical Couette flow, predicting unusual effects upon the addition of drag-reducing chemicals. The generalizability of the method to other turbulent processes is discussed in the conclusion.

2. Turbulent Poiseuille channel flow and everywhere positive functions

The symbols used here for position (x, y, θ, ϕ) and velocity (u, w, U) in channel flow are indicated in figure 2. Although the mean velocity $U = U(z)$ is observed to have no inflexions, the instantaneous downstream velocity $u(\mathbf{x}, t)$ has frequent inflexions. The horizontal average of the equations of motion for a normal fluid of constant density is written as

$$\left(-\nu\frac{d^2U}{dz^2}\right) = \frac{\tau_0}{z_0} + \left(-\frac{d\overline{wu}}{dz}\right), \tag{2.1}$$

where

$$\tau_0 = \left(-\frac{1}{\rho}\frac{d\overline{P}_0}{dx}\right) z_0 \equiv U_\tau^2, \tag{2.2}$$

ρ is the density, $\overline{P}_0$ the average pressure and $\overline{wu}$ the Reynolds stress. The first integral of (2.1) is

$$\tau = \frac{\tau_0}{z_0} z = \overline{wu} - \nu\frac{dU}{dz}. \tag{2.3}$$

The sum of the Reynolds and viscous stress varies linearly with z, both vanishing at $z = 0$ for the symmetric mean flow indicated in figure 2.

At this point a representation of the curvature of U is introduced to reflect the presumed inflexional instability mechanism and the observation that the curvature is of one sign. One writes

$$\left(-\frac{z_0^2}{U_\tau}\frac{d^2U}{dz^2}\right) \equiv I^*I, \tag{2.4}$$

where

$$I(\phi) = \sum_{k=0}^{\infty} I_k e^{ik\phi}, \tag{2.5}$$

the superscript star denotes the complex conjugate, and

$$\phi = \pi(z+z_0)/z_0, \quad 0 \leqslant \phi \leqslant 2\pi \quad (2\theta \equiv \phi - \pi). \tag{2.6}$$

It was shown by Fejér (1916), and we shall see shortly, that the half-space summation over $0 \leqslant k \leqslant \infty$ in (2.5) provides a complete representation of an everywhere positive function.

The boundary conditions on the velocity field are in general that

$$w = u = 0 \quad \text{on} \quad \phi = 0, 2\pi \tag{2.7a}$$

and for a smooth boundary

$$\partial w/\partial z = 0 \quad \text{on} \quad \phi = 0, 2\pi. \tag{2.7b}$$

Hence

$$(\overline{wu})_{\phi=0,2\pi} = \left(\frac{d}{dz}\overline{wu}\right)_{\phi=0,2\pi} = 0 = \left(\frac{d^2}{dz^2}\overline{wu}\right)_{\phi=0,2\pi} \tag{2.8}$$

and from (2.1), (2.4) and (2.8)

$$(I^*I)_{\phi=0,2\pi} = U_\tau z_0/\nu \equiv R_\tau, \tag{2.9}$$

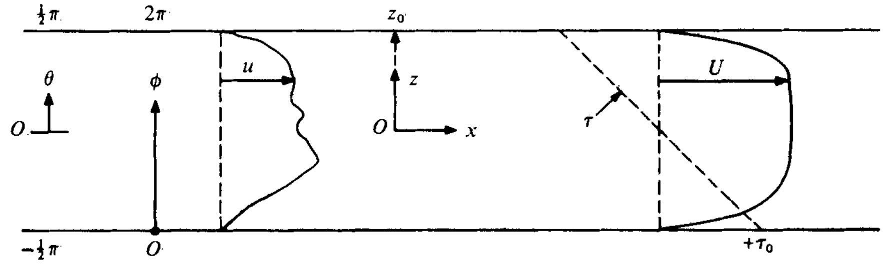

FIGURE 2. Turbulent Poiseuille flow in a channel.

where R_τ is the stress (or pressure) Reynolds number, which may be presumed given as the experimental parameter for this flow.

The relation between the representation for I in (2.5) and a normal Fourier representation for $I*I$ can be established straightforwardly. Let

$$I_m = A_m + iB_m, \tag{2.10}$$

where the A_m and B_m are all real. Then

$$\begin{aligned}(I*I)(\phi) = \sum_{k=0}^{\infty} (2-\delta_{k,0}) \sum_{m=0}^{\infty} (A_m A_{m+k} + B_m B_{m+k}) \cos k\phi \\ + \sum_{k=1}^{\infty} 2 \sum_{m=0}^{\infty} (A_{m+k} B_m - A_m B_{m+k}) \sin k\phi.\end{aligned} \tag{2.11}$$

For symmetric $(I*I)(\phi)$ one may write

$$\left.\begin{aligned}(I*I)(\phi) \equiv \sum_{m=0}^{\infty} C_k \cos k\phi, \\ C_k = (2-\delta_{k,0}) \sum_{m=0}^{\infty} I_m I_{m+k}, \quad I_m \,\text{real}.\end{aligned}\right\} \tag{2.12}$$

Although there are as many amplitudes I_k as Fourier coefficients C_k, they are related through (2.12) in a quadratic fashion. Hence, for a given set of n (uniquely defined) coefficients C_k, there are generally n^2 sets $\{I_k\}$ which satisfy (2.12). Fejér's (1916) paper was a first formal study on the nature of this redundancy. Work in progress suggests that the I_k sets approach a narrow and smooth distribution in amplitude, k space for those particular values of C_k near functions which are marginally positive.

The next section describes the qualitative features of the velocity profiles U which emerge as a consequence of a hypothesized 'smooth' spectrum for I_k.

3. A 'smoothness' hypothesis

A mathematical representation of a process is a particularly appropriate one if a simple aspect of that representation reflects a principal observed consequence of that process. In quasi-linear processes, a single eigenfunction of the linear theory often describes quite closely the observed qualitative behaviour of a system. However, it is quite clear that a single Fourier component, or a single element of I in (2.5), fails to describe observed nonlinear features of shear-flow turbulence. Here one explores the possibility that some simple integral feature of the representation is related to observations. Indeed, the experimental description (Willmarth & Bogar 1977) of an intense small-scale vortical motion, expanding in scale, changing shape and rising through the boundary layer, suggests a rather uniform projection of the momentum transport process on all scales. Hence one seeks the integral consequences of an I_k spectrum whose features are unimportant in some asymptotic sense. Of course, at some very small scale, say k_ν, one expects viscous effects to reduce I_{k_ν} to a vanishingly small value. Yet the change in I_k might be 'smooth' in the range of k where I_k is finite.

To explore the consequence of this possibility, it is convenient to sum (2.5) by parts, assuming that there is some k_ν such that $I_k \simeq 0$ for $k > k_\nu$. One writes

$$I(\phi) = \sum_{k=0}^{\infty} I_k e^{ik\phi} \simeq \sum_{k=0}^{k_\nu} I_k e^{ik\phi}, \tag{3.1}$$

and defines

$$(\Delta I)_k = I_{k+1} - I_k, \quad (\Delta^2 I)_k = (\Delta I)_{k+1} - (\Delta I)_k, \quad F_k = (e^{ik\phi}-1)/(e^{i\phi}-1).$$

Hence $(\Delta F)_k = e^{ik\phi}$ and

$$\sum_{k=0}^{k_\nu} I_k e^{ik\phi} = \sum_{k=0}^{k_\nu} I_k (\Delta F)_k = I_k F_k \Big|_0^{k_\nu+1} - \sum_{k=0}^{k_\nu} (\Delta I)_k F_{k+1},$$

or

$$\sum_{k=0}^{k_\nu} I_k e^{ik\phi} = -\frac{I_0}{e^{i\phi}-1} - \frac{e^{i\phi}}{e^{i\phi}-1} \sum_{k=0}^{k_\nu} (\Delta I)_k e^{ik\phi}. \tag{3.2}$$

Repeating this summation by parts on the final sum in (3.2), one may write

$$\sum_{k=0}^{k_\nu} I_k e^{ik\phi} = \frac{1}{1-e^{i\phi}} \left[I_0 + \frac{e^{i\phi}}{1-e^{i\phi}} \left\{ (\Delta I)_0 + e^{i\phi} \sum_{k=0}^{k_\nu} (\Delta^2 I)_k e^{ik\phi} \right\} \right]. \tag{3.3}$$

A general expansion in terms of higher differences follows, but will not be used here.

One now observes that, if I_k is 'smooth' in the sense that

$$(\Delta I)_k = O(I_0/k_\nu), \quad (\Delta^2 I)_k = O(I_0/k_\nu^2) \tag{3.4}$$

then from (3.3)

$$I(\phi) \simeq \sum_{k=0}^{k_\nu} I_k e^{ik\phi} = \frac{I_0}{1-e^{i\phi}} + O\left(\frac{I_0}{k_\nu}\right) \tag{3.5}$$

for all angles $\phi \gg k_\nu^{-1}$. Hence a unique, and surprisingly simple, form for $I(\phi \gg k_\nu^{-1})$ exists if the weak condition (3.4) is met.

From (2.4) and (3.5) a complete 'velocity-defect law' can be determined as

$$(U_m - U)/U_\tau = \pi^{-2} |I_0|^2 \ln \operatorname{cosec} \tfrac{1}{2}\phi. \tag{3.6}$$

This form is parabolic in mid-regions of the flow and logarithmic near the boundary. For $I_0 = O(1)$, it is the only law whose qualitative behaviour is insensitive to the features of the underlying spectrum, yet reflects the instability mechanisms which are presumed responsible for the positive curvature.

The velocity-defect law (3.6) contains no hint of the complicated double logarithmic structure seen in figure 1. Clearly, the hypothesis of spectral 'smoothness' must break down at large k when drag-reducing chemicals are added to the fluid. Models of such a process, and of the normal termination of the I_k spectrum, are discussed in the following section (§4).

At the other end of the k spectrum comparison of the defect law (3.6) and data from *Poiseuille*-like flow with various pressure gradients indicates the dependence of mid-region amplitudes on those pressure gradients. Since it is the process behind $\overline{wu}$, the Reynolds stress, which one has presumed to lead to spectral 'smoothness' for I, it is plausible that other effects contributing to I could disrupt smoothness. From (2.1) and (2.4) one sees that the mean pressure gradient as well as $d\overline{wu}/dz$ determines I^*I. Hence a non-smooth contribution to the spectrum at $k = 0$ occurs and is discussed in §5.

The proposal that 'smoothness', in the sense of (3.4), could lead to a unique velocity-defect law and the form (3.6) of that law have both been reported before (Malkus 1961) and discussed in the literature (Townsend 1961). However, that earlier work was done in ignorance of Fejér's study, and the correct summation limits in (2.5). In consequence, the amplitude of U deduced in that work depended entirely upon the unknown amplitude of a postulated smallest scale of motion. Note that the amplitude of U found here [see (3.6)] depends on I_0, a quantity represented in every Fourier component (2.11) of the Reynolds stress.

4. Spectral tails and the ultimate profile

The assumption of spectral smoothness may seem less plausible at those wavenumbers where viscous effects first become as important as nonlinear advection, for it is at this scale of motion that the smallest finite amplitude instabilities are observed to occur. Also, it is just at these wavenumbers that drag-reducing chemical additives are presumed to be most effective in producing non-Newtonian effects. In this section, spectral tails are modelled in order to investigate conditions under which altered mean flow profiles will occur. A principal goal is to determine those aspects of any profile alteration which are insensitive to spectral details of the model.

A plausible requirement for a fluid-dynamical spectral tail is that it should drop off faster than any finite negative power of k, in order that all moments of the flow be finite. A second requirement is that all presumed tails are continuous with and match the smoothness condition at the wavenumber where the tail joins the inertially controlled lower wavenumber spectrum. The simplest tail to meet these requirements is a modified exponential. Hence it is proposed that we explore the consequences of the tail

$$I_{k>k_0} = I_{k_0}[1+\alpha(k-k_0)+\beta(k-k_0)^2+\ldots]\exp[-\gamma(k-k_0)], \tag{4.1}$$

which is continuous with the spectrum at $k \leqslant k_0$ and for which

$$\left.\begin{aligned}(\Delta I)_{k_0} &= I_{k_0}[(1+\alpha+\beta+\ldots)e^{-\gamma}-1],\\ (\Delta^2 I)_{k_0} &= I_{k_0}[(1+2\alpha+4\beta+\ldots)e^{-2\gamma}-2(1+\alpha+\beta+\ldots)e^{-\gamma}+1].\end{aligned}\right\} \tag{4.2}$$

In (4.1), $\alpha, \beta, \gamma, \ldots$ are arbitrary constants subject only to the condition to be imposed on $(\Delta I)_{k_0}$ and $(\Delta^2 I)_{k_0}$. The wavenumber k_0 ($< k_\nu$) marks the low wavenumber end of the tail, and γ characterizes the degree of abruptness of the spectral cut-off.

The general spectrum (2.5) determining the profile curvature may now be written with the help of (4.1) as

$$I(\phi) = \sum_{k=0}^{k_0} I_k e^{ik\phi} + \sum_{k=k_0+1}^{\infty} I_{k_0}[1+\alpha(k-k_0)+\beta(k-k_0)^2+\ldots]\exp[-\gamma(k-k_0)]\exp(ik\phi), \tag{4.3}$$

or, by performing the infinite sum over the tail, as

$$I(\phi) = \sum_{k=0}^{k_0} I_k e^{ik\phi} + I_{k_0} \exp[i(k_0+1)\phi] \exp(-\gamma) \left[\frac{1}{1-e^{-a}} + \alpha \frac{1}{(i-e^{-a})^2} + \beta \frac{1+e^{-a}}{(1-e^{-a})^3} + \dots\right], \quad (4.4)$$

where $a = \gamma - i\phi$.

As it stands, with k_0 and all the I_k unspecified, (4.4) can describe any plausible turbulent profile of positive curvature at any Reynolds number. At this point one seeks asymptotic consequences of the smoothness hypothesis

$$(\Delta I)_k = O(I_0/k_0), \quad (\Delta^2 I)_k = O(I_0/k_0^2), \quad 0 \leqslant k \leqslant k_0, \quad (4.5)$$

and from (3.2), (3.3) and (4.4) concludes that for $\phi \gg k_0^{-1}$

$$I(\phi) = \frac{1}{1-e^{i\phi}} \left[I_0 - I_{k_0} \exp[i(k_0+1)\phi] \left\{\frac{1-e^{-\gamma}}{1-e^{-a}} + \alpha \frac{e^{-a}-e^{-\gamma}}{(1-e^{-a})^2} + \beta \frac{(e^{-a}-e^{-\gamma})(1+e^{-a})}{(1-e^{-a})^3} + \dots\right\}\right] + O\left(\frac{I_0}{k_0}\right). \quad (4.6)$$

If then $\alpha, \beta, \gamma = O(k_0^{-1})$, (4.6) reduces to the result given in (3.5) and to the velocity-defect law (3.6).

The unanticipated aspect of the mean velocity profiles determined by (4.6) is the double logarithmic structure. This is most easily seen from the leading term of the inner bracketed expression in (4.6) for $\phi, \alpha, \beta, \gamma, \dots \ll 1$. One writes

$$\{\ \} \simeq \left\{\frac{\gamma}{\gamma - i\phi} + \alpha \frac{i\phi}{(y-i\phi)^2} + 2\beta \frac{i\phi}{(\gamma - i\phi)^3} + \dots\right\}. \quad (4.7)$$

Hence for $\gamma \gg \phi \gg k_0^{-1}$, the bracket (4.7) approaches unity and the resulting logarithmic velocity profile from (2.4) and (4.6) has amplitude $|I_0|^2 + |I_{k_0}|^2$. In contrast, when $\phi \gg \gamma, \alpha, \beta, \dots$, the bracket (4.7) approaches zero and the resulting logarithmic velocity profile from (2.4) and (4.6) has the 'outer' amplitude $|I_0|^2$.

The precise form of the change from one logarithmic law to the other depends on $\alpha, \beta, \gamma, \dots$. For example, if only α, β and γ are much greater than k_0^{-1}, and the 'smoothness' conditions (4.5) are used to relate α and β to γ through (4.2), then for $\gamma \ll 1$ bracket (4.7) may be written

$$\{\ \} \simeq \{\gamma^2/(\gamma - i\phi)^2\}. \quad (4.8)$$

The transition from the 'inner' logarithmic law of amplitude $|I_0|^2 + |I_{k_0}|^2$ to the 'outer' logarithmic law of amplitude $|I_0|^2$ proceeds in ϕ as $\{\ \}^*\{\ \}$, and hence is 80 % complete within a factor of three on either side of $\phi = \gamma$. If the term beyond β in (4.1) has a value large compared with k_0^{-1}, then the term added to bracket (4.8) has an effect of higher order in γ on the abruptness of the transition.

The data used by Virk (1975) in constructing the idealization shown in figure 1 in this paper are reproduced here as figure 3. The symbol S_f in the figure indicates the fractional drag reduction. A variety of different drag-reducing chemicals were used in those flows by a listed number of different observers. More recent data obtained by Reischman & Tiederman (1975) (also data in Frenkiel, Landahl & Lumley 1977) have increased the scatter seen in figure 2, but retain the features of an 'ultimate' profile independent of the type or amount of drag-reducing chemical and a point of transition

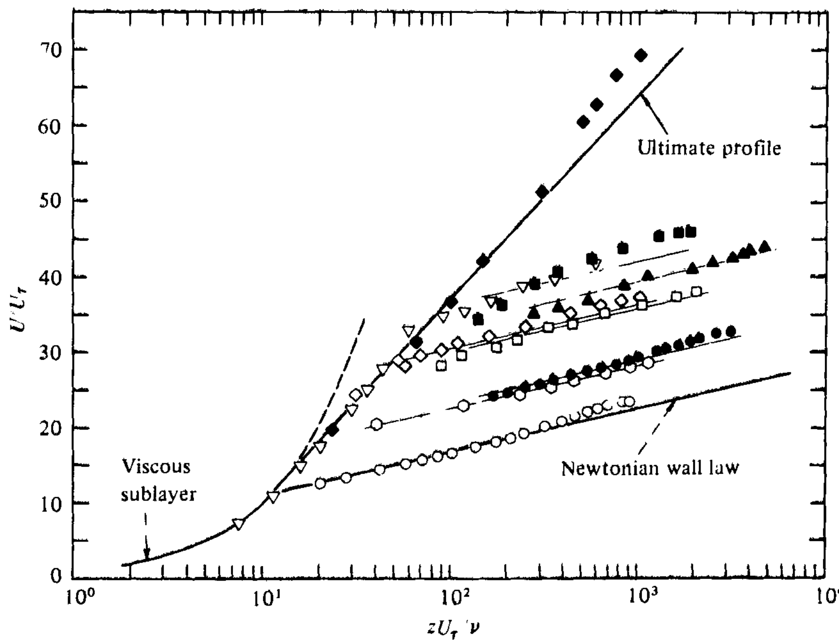

FIGURE 3. Mean velocity profiles resulting from various drag-reducing chemical additives. (After Virk 1975.)

	▲	□	■	○	◇	◆	▽	○	●
S_F	0·74	0·64	0·94	0·27	0·70	1·59	~ 1·0	0·0	0·29
Source	Elata	Goren	Goren	Patterson	Rollin	Rollin	Rudd	Virk	Virk

to an outer Newtonian velocity profile fully dependent on the type and amount of additive. It would appear that all these data can be described by a smooth spectrum for I and the three spectral parameters $|I_0|$, $|I_{k_0}|$ and γ.

The data suggest that $|I_0|$ is approximately constant, and is inversely proportional to the square root of the accepted value for von Kármán's constant. The value of $|I_{k_0}|$ is determined from the ratio of the slopes of the 'ultimate' profile and the 'normal' Newtonian wall law. It also appears to be approximately constant for all reported flows containing polymer additives. The ratio

$$|I_{k_0}|/|I_0| = 2 \pm 0{\cdot}15 \tag{4.9}$$

from the Virk data, but is more nearly 1·5 from the Reischman & Tiederman data. In the spectral tail discussed here, the exponential decay factor γ alone determines the point of transition, e.g. (4.8). Hence the data suggest that γ is determined by the amount and type of chemical additive and can approach a value of unity.

There are many other kinds of additives to shear flow which cause different effects from those caused by the polymer chemical additives used in the flows reported on in figure 3. For example, sand or glass beads near the boundary lead to a gravitational stabilization perhaps best dealt with as a two-fluid process. Additives such as neutrally buoyant sticks or rods which are large compared with the smallest unstable scales of motion plausibly produce a viscous stabilization which could alter the value of $|I_{k_0}|$ and the inertially produced spectral smoothness in a range of $k < k_0$. Polymer additives, when expanded, perhaps can be thought of as neutrally buoyant rods also, but rods which are small compared with the smallest scales of motion. The available data are rationalized (e.g. Hinch 1977; Bark & Tinoco 1978) by suggesting how the polymer additives could inhibit instabilities and abruptly truncate the I_k spectrum at a particular stress level. It should be noted that a sharp truncation of the I_k spectrum does not necessarily imply a sharp truncation of the kinetic energy spectrum, for a primary effect of the polymer additive could be the high wavenumber decorrelation of the Reynolds stress $\overline{uw}$. That the amplitude $|I_{k_0}|$ is observed to remain more or less constant, independent of the concentration of polymer additive, indicates the strong inertial control of the spectrum right up to viscous limit.

A mechanistic determination of $|I_{k_0}|$ may not be immediately practical. Presuming, as this author does, that Landahl's stability studies are indeed in the correct direction, then a quantitative understanding is required of how those instabilities determine the amplitude of the 'travelling inflexions' which in turn initiate later instabilities. However, an estimate of amplitude can be found in terms of the boundary Reynolds number and maximum profile curvature. It is observed that at $R_B = U_\tau z/\nu \simeq 8$ the mean velocity profile departs from the laminar viscous form it has nearer the boundary. This value is found for normal fluids as well as for fluids exhibiting large drag reduction due to chemical additives. It would appear that additives which do not alter the normal viscosity of the fluid do not alter this critical Reynolds number for instability, but rather reduce the number and modify the momentum transporting properties of the growing disturbances. The data indicate for normal fluids that the Reynolds stress transports half the momentum by $R_B \simeq 10$. The corresponding value for flows exhibiting significant drag reduction is $R_B \simeq 14$ (Reischman & Tiederman 1975, figure 8). From the same data one observes that the curvature of the mean profile at the above values of R_B is near its maximum, with a ratio to the curvature at the boundary of approximately $\frac{1}{4}R_\tau$ for normal flow and approximately $\frac{1}{8}R_\tau$ for drag-reduced flows. Parenthetically, one notes that the process causing the abruptness of spectral truncation which permits an ultimate profile to occupy the entire flow must, of course, be extremely non-Newtonian. However, from the R_B's above it is equally startling that the inertial processes determining the amplitude of the smallest momentum transporting scales are so little affected by, or so able to compensate for, this non-Newtonian behaviour.

For an I_k spectrum abruptly truncated at k_0, the first contribution to profile curvature near the boundary is of scale $z_0 k_0^{-1}$. If this scale is identified with the boundary-layer scale determined by R_B above, then

$$k_0 \simeq R_\tau / R_B. \tag{4.10}$$

Hence an estimate can be made of a representative product amplitude $|I_0|\,|I_{k_0}|$ of the smoothest truncated spectrum consistent with the boundary condition (2.9). Using (2.4) and (2.12) one finds

$$-\nu \frac{d^2U}{dz^2} = \frac{\nu U_\tau}{z_0^2}(I^*I)_{\max} < \frac{\nu U_\tau}{z_0^2}\frac{k_0^2}{4}|I_0|\,|I_{k_0}|. \tag{4.11}$$

From (4.10), (4.11) and the observed maximum curvature it follows that

$$|I_0|\,|I_{k_0}| = O(R_B^2). \tag{4.12}$$

The explicit implication of (4.12) is that the amplitudes of the logarithmic mean velocity profiles are determined by the (viscous) stability constraints on the smallest scales of motion. However, the ratio $|I_{k_0}|/|I_0|$, observed to be approximately two, is not determined here and appears to be a complicated consequence of the interaction of large and small momentum transporting scales. Aspects of the large-scale process are discussed in the following section.

5. Pressure-gradient effects, plane Couette flow and cylindrical Couette flow

The simple defect law (3.6) results from an assumption of spectral smoothness which does not take into account, from (2.1), the contribution to profile curvature due to a pressure gradient. The profile curvature at the boundaries is entirely determined by the pressure gradient, of course, for there the Reynolds stress and its derivative vanish. In the steady state, the mean pressure gradient is constant across the entire channel and hence uniformly produces profile curvature. The advection term $d\overline{wu}/dz$ fails by a small amount in mid-flow to cancel that uniform curvature production. It is proposed to model this process by adding an arbitrary amplitude at $k = 0$ to an otherwise smooth spectrum I_k.

If then the smoothness assumption (3.4) is appropriate for the entire spectrum except $(\Delta I)_0$ and $(\Delta^2 I)_0$, one can write (3.3) as

$$I(\phi) = (I_0 - I_1) + \frac{I_1}{1-e^{i\phi}} + O\left(\frac{I_0}{k_\nu}\right) \tag{5.1}$$

for all $\phi \gg k_\nu^{-1}$. From (2.4) and (5.1) the 'pressure-modified' symmetric velocity-defect law is found to be

$$\frac{U_m - U}{U_\tau} = \frac{1}{\pi^2}\left[I_1^2 \ln \operatorname{cosec}\frac{\phi}{2} + \frac{I_0(I_0-I_1)}{2}(\pi-\phi)^2\right]. \tag{5.2}$$

Equation (5.2) describes mean speed data for turbulent Poiseuille flow in channels within the limits of experimental error. With I_1^2 chosen to be inversely proportional to von Kármán's constant, $(I_0 - I_1)/I_1$ is approximately 0·2 for moderate to moderately high Reynolds numbers. Channel data at high and very high Reynolds numbers which might determine a possible weak dependence of $(\Delta I)_0$ on R_τ do not exist.

Plane turbulent Couette flow provides quite different challenges of interpretation. While there is no pressure gradient in plane Couette flow, the curvature of the mean profile must change sign in the middle of the flow. This one profile inflexion is inertially stable because of its sign, yet the formalism used here for positive curvatures is of use in only half of the channel. In lieu of a more general formulation, asymptotic consequences of spectral smoothness in I_k for the half-channel are given below, subject to $I(\pi) = 0$ so that the antisymmetry conditions can be met. As for (3.5), but subject to $I(\pi) = 0$, one finds for $\phi \gg k_\nu^{-1}$

$$I(\phi) = I_0\frac{1+e^{i\phi}}{1-e^{i\phi}} + O\left(\frac{I_0}{k_\nu}\right), \tag{5.3}$$

or

$$I^*I(\theta) = |I_0|^2 \tan^2\theta + O\left(\frac{I_0^2}{k_\nu}\right) \tag{5.4}$$

in terms of the angle $2\theta = \phi - \pi$ measured from channel centre. Hence the Couette half-channel velocity-defect law from (5.4) and (2.4) is written as

$$U(\theta)/U_\tau = 4\pi^{-2}|I_0|^2[\ln\sec\theta - \tfrac{1}{2}\theta^2] + C\theta, \tag{5.5}$$

where C is an unknown constant of integration determining the slope of the mean profile in mid-flow. It is this appearance of the constant C which suggests that Couette flow may have no normal defect law and possibly could be quite different from pressure-driven flows. Indeed, the evidence of Reichart (1959) suggests that CU_τ is a constant and that approximately 20 % of the velocity difference between the moving boundaries occurs in the interior of the fluid. Also, the upper-bound theory for Couette momentum transport of Busse (1970) derives a C in keeping with the Reichart data. Plane Couette flow is the epitome of simple shear flows, the realization of the constant stress layer, and deserves to be a central testing ground of shear turbulence theory. One hopes that plane Couette turbulence will receive more experimental attention at high Reynolds numbers in order to resolve the apparent conflicts between the data and traditional theories.

An experimental study of Couette flow between rotating cylinders, in the narrow-gap and high Reynolds number limit, one day may provide the additional data sought in the preceding discussion. However, it is a difficult experiment because the centrifugal effects, either stabilizing or destabilizing and measured by the Taylor number (e.g. Chandrasekhar 1961, p. 296), are always present to some extent. Perhaps as interesting and experimentally a more accessible limit is the narrow-gap, high Taylor number extreme of cylindrical Couette flow. In this limit the inviscid destabilization is caused principally by the radial gradient of Ur, where U is the mean zonal velocity and r is the distance of the fluid element from the centre of rotation. In the observed turbulent flows (e.g. Taylor 1936), the gradient of Ur is of one sign only. Hence, in this limit and under the hypothesis of spectral 'smoothness', (3.5) times its complex conjugate would describe the qualitative character of that gradient. One predicts then that U varies as $I_0^2 \tan\theta$ across the gap $-\frac{1}{2}\pi \leqslant \theta \leqslant +\frac{1}{2}\pi$. The limited data are compatible with this result; however, the observed amplitude I_0^2 is so small that the entire effect is confined to a region very near the boundary. Although Millikan's (1938) argument applies here, no logarithmic region has been reported for this flow. The conditions under which centrifugal destablization dominates over inflexional destabilization in this turbulent flow are yet to be established (e.g. Van Atta 1966). In gathering these additional data it is proposed that one should also employ drag-reducing chemicals. [A weak axial flow in the narrow gap could maintain statistical homogeneity and un-degraded additives.] As a consequence, an 'inner' $\tan\theta$ region of greater amplitude is anticipated, paralleling the 'ultimate profile' of §4, and reflecting the effects of spectral truncation at high wavenumbers.

6. Conclusion

The observed positive curvature of the mean velocity in turbulent channel flow is presumed to have its mechanistic origin in the brief and violent instabilities which are the principal agents of the momentum transfer process. The representation of the mean curvature as an everywhere positive spectral function sets the stage for the smoothness hypothesis of §3, whose asymptotic consequence is an explicit velocity-defect law.

This stability-oriented representational framework for nonlinear turbulent equilibration suggests new theoretical and physical objects of study. In §4, it is shown that a spectral tail, whose extent is restricted by drag-reducing chemical additives, gives rise to an inner logarithmic layer of the mean velocity, whose slope is larger than the slope of the normal logarithmic layer.

It is not proposed that this approach can lead to detailed understanding of the time-dependent dynamics of turbulence, but rather that it isolates certain amechanistic consequences of the finite amplitude stabilizing process. It is in the latter direction that generalization is possible. Shear turbulence in different geometries should lead to qualitatively different average flows. The example of Couette turbulence is discussed in the preceding section. In another example, axial turbulent flow between two cylinders departs significantly from a logarithmic profile when the inner cylinder is sufficiently small. With the approach taken in §3, application of the appropriate boundary and symmetry conditions can lead to an asymptotic defect law.

The principal arena of generalization involves flows with quite different inviscid stability criteria. The example of laboratory thermal convection has been studied in the past in a quantitative context on the basis of the additional hypothesis of maximum heat transport (Malkus 1961) and the thermal profile found to compare favourably with the data of Townsend (1959). That theoretical study would be strengthened and its assumptions reduced in number by a mathematical restatement using the formalism outlined here. Stratified shear flows and convection with a mean shear both have observed turbulent mean profiles of temperature and velocity for which one could seek amechanistic defect laws from the appropriate positive-definite stability functions. Such results could be compared with the less than satisfactory (Sundararajan & Macklin 1976) dimensional and mixing-length theories for such flows.

Finally, it is understood that the implicit rule in presenting any deduction of established results in an unfamiliar manner is that the new manner should require fewer assumptions, be more generalizable and not contradict the tested premises of the earlier work. It is hoped that this study may meet those requirements.

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